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A comparison between models of thermal fields in laser and electron beam surface processing

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Abstract—In laser and electron beam surface processing the knowledge of the thermal field and, particularly, of the maximum temperatures attained is a very important issue. The two-dimensional solution for a uniform strip heat source moving at a constant velocity along the surface of a semi-infinite body is analysed. A comparison between the approximate one-dimensional and the two-dimensional models is presented. The ratio between the maximum temperature values is given by the two models and the time at which they are reached are reported. Guidelines for estimating the range of variables within which the results of the two models are in a reasonable agreement are given.

INTRODUCTION

THE THEORETICAL determination of the temperature field in a semi-infinite solid heated by a moving heat source is a well-known problem in heat conduction and various solutions are available for different source geometries [1–7]. Although the aforementioned problem may be encountered in many applications, it is typical in laser and electron beam surface transformation hardening of metallic materials. In these processes a suitably shaped beam rapidly heats only a thin layer of the material surface; then the heat flows towards the cool bulk by conduction and the surface quench occurs [8]. The layer to be hardened (typically 0.1–1 mm) must reach a temperature greater than the material dynamic austenite transformation temperature. Thus, the fundamental purpose of the heat conduction analysis is to estimate the maximum temperature attained as a function of the depth.

For a uniform strip heat source moving at a constant velocity along the solid surface, in some conditions, the quasi-steady state two-dimensional (2-D) exact solution can be approximated by the one-dimensional (1-D) semi-infinite solution with a stationary constant source impinging for an amount of time equal to the interaction time of the actual moving source [9]. The latter, simpler, solution can be profitably used provided the heat diffusion length is small in comparison with the strip width. Nevertheless, under many circumstances a better accuracy is required and therefore a deeper insight into the differences between the two models is useful.

La Rocca [9] gives a detailed analysis of the 1-D models for both a semi-infinite body and a finite slab. In ref. [9] new dimensionless physically meaningful parameters are introduced and the range of applicability of the models are extensively investigated.

Cline and Anthony [5] present the analytical solution for a Gaussian source moving at a constant velocity and correlate the cooling rate distribution and the depth of melting with the size, the velocity and the power level of the spot. However, they make no comparison with the results of simpler models.

Still with reference to a Gaussian moving source, Chen and Lee [6] demonstrate the existence of a critical velocity below which the effects of the source motion become practically negligible and the temperature field is reasonably coincident with the one due to a stationary source.

Sanders [7] presents the conditions under which the solution for a Gaussian moving spot as a function of its normalized velocity can be used. At low speed his solution approximates the one obtained by Lax [10] for a stationary spot while at high speed it approaches the 1-D solution.

To the authors' knowledge, no accurate comparison between the 1- and 2-D solution has yet been made. In ref. [11] 2-D effects are said to become noticeable when the ratio of the spot velocity to the rate of heat diffusion in the solid is less than a given value (3.5). Moreover, the effects of depth are not considered.

In this paper 1- and 2-D solutions are compared and, for any depth, both the ratio of the maximum temperatures evaluated by the two solutions and the time at which they occur are provided. Results are presented as a function of dimensionless parameters in diagrams and tables.

NOMENCLATURE

b	hot spot half-width
B	dimensionless parameter, equations (8a)
k	thermal conductivity
K_0	modified Bessel function of the second kind, order zero
l	hot spot length in ref. [2]
q	heat flux
R	dimensionless ratio, equation (13)
t	time
T	temperature
T^+	dimensionless temperature, equation (8b)
v	heat source velocity
$\mathbf{v}$	heat source vectorial velocity
x, z	fixed Cartesian coordinates
x^*, z^*	moving Cartesian coordinates
X, Z	dimensionless Cartesian coordinates, equations (8a).

Greek symbols

α	thermal diffusivity
β	variable, equations (A2)
γ	variable, equations (A2)
ζ	variable, equations (8a)
θ	time, equations (1)
μ	variable, equation (5)
ξ	dimensionless time, equations (8a)
$\bar{\xi}$	roots of equations (17) and (19)
τ	dwell time.

Subscripts

1-D	one-dimensional
2-D	two-dimensional.

GEOMETRY AND MATHEMATICAL DESCRIPTION

Consider a strip heat source, moving at a constant relative velocity $\mathbf{v}$ over the surface of a semi-infinite body. Geometry and coordinates are schematically shown in Fig. 1. The body is isotropic and homogeneous; it is conveniently described by using the fixed rectangular coordinates x and z. The surface $z = 0$ is thermally insulated except over the region $vt-b \leqslant x \leqslant b+vt$ at time t, where a constant uniform heat flux is postulated, q_0, with $2b$ the width of the hot strip. From the moving heat source theory [3], by a Galilean coordinates transform

$$x^* = x - vt; \quad z^* = z; \quad \theta = t \tag{1}$$

a mathematical statement of the problem is the solution of

$$\frac{\partial^2 T}{\partial x^{*2}} + \frac{\partial^2 T}{\partial z^{*2}} = -\frac{v}{\alpha}\frac{\partial T}{\partial x^*} + \frac{1}{\alpha}\frac{\partial T}{\partial Q}$$

$$\text{for } |x^*| < \infty, \quad 0 \leqslant z^* < \infty, \quad \theta > 0 \tag{2a}$$

$$T(x^*, z^*, 0) = 0 \quad \text{for } |x^*| < \infty, \quad 0 \leqslant z^* < \infty \tag{2b}$$

$$-k\frac{\partial T(x^*, 0, \theta)}{\partial z^*} = q(x^*)$$

$$= \begin{cases} q_0 & \text{for } |x^*| \leqslant b \\ 0 & \text{for } |x^*| > b\text{'} \end{cases} \quad \theta > 0 \tag{2c}$$

$$T(x^*, z^*, \theta) = 0 \quad \text{for } x^* \to \pm\infty, \quad z^* \to \infty, \quad \theta > 0. \tag{2d}$$

With initial condition (2b), values of T are the temperature rise. Properties k and α are assumed to be independent of temperature and position.

The problem may be stated in a simple way by neglecting the heat flow along the x-axis. Consider the geometry shown in Fig. 2, where now the heat flux over the semi-infinite body is a uniform step function of time. If τ is the dwell time, that is the amount of time the spot on the surface is exposed to the uniform and constant heat flux, a mathematical statement of the problem is the solution of

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3a}$$

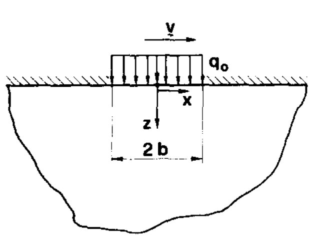

FIG. 1. Semi-infinite body heated over the surface by a moving heat source $2b$ wide.

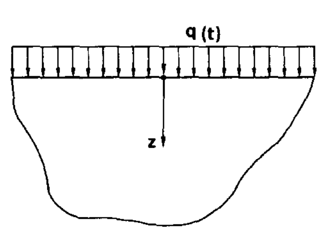

FIG. 2. Semi-infinite body heated over the surface with a uniform heat flux for a finite time amount.

$$T(z,0) = 0 \qquad \text{for } 0 \leqslant z < \infty \tag{3b}$$

$$-k\frac{\partial T(0,t)}{\partial z} = q(t) = \begin{cases} q_0 & \text{for } 0 \leqslant t \leqslant \tau \\ 0 & \text{for } t > \tau \end{cases} \tag{3c}$$

$$T(z,t) \to 0 \qquad \text{for } z \to \infty, t > 0 \tag{3d}$$

where T is still the temperature rise and properties k and α are independent of temperature and position.

In the following the differences between the solutions to problems (2) and (3) will be evaluated and suggestions for the use of the 1-D solution instead of the 2-D solution will be given.

AVAILABLE SOLUTIONS

The exact solution to the first problem for a semi-infinite solid is given by Munari [12], in quasi-stationary ($\theta \to \infty$) conditions and for $l \to \infty$

$$T(x^*,z^*) = \frac{q_0\alpha}{k\pi^{1/2}} \lim_{\theta\to\infty} \int_0^\theta \frac{\exp[-(z^{*2}/4\alpha(\theta-\theta'))]}{[4\alpha(\theta-\theta')]^{1/2}} \times \left\{\operatorname{erf}\left[\frac{x^*+b+v(\theta-\theta')}{(4\alpha(\theta-\theta'))^{1/2}}\right] - \operatorname{erf}\left[\frac{x^*-b+v(\theta-\theta')}{(4\alpha(\theta-\theta'))^{1/2}}\right]\right\} d\theta'. \tag{4}$$

Expression (4) can also be derived taking to the limit ($l \to \infty$) the solution given by Carslaw and Jaeger (p. 270 in ref. [2]) for an infinite solid and is equivalent to the solution for a semi-infinite solid given by Carslaw and Jaeger (p. 269 in ref. [2]). In view of the following comparison, solution (4) is now written with reference to the fixed coordinates system. Let

$$\mu = \theta - \theta' \tag{5}$$

then expression (4) becomes

$$T(x,z,t) = \frac{q_0\alpha^{1/2}}{2k\pi^{1/2}} \int_0^\infty \exp\left(-\frac{z^2}{4\alpha\mu}\right) \times \left\{\operatorname{erf}\left[\frac{v}{(2\alpha)^{1/2}} \frac{(x+b)/v-t+\mu}{(2\mu)^{1/2}}\right] - \operatorname{erf}\left[\frac{v}{(2\alpha)^{1/2}} \frac{(x-b)/v-t+\mu}{(2\mu)^{1/2}}\right]\right\} \frac{d\mu}{\mu^{1/2}}. \tag{6}$$

Note that, for a given z, to the quasi-stationary temperature field in the moving coordinates system correspond equal temperature profiles, for each x, in the fixed coordinates system. Furthermore, if the dwell time of the hot spot over a point of the surface of the solid can be expressed as

$$\tau = 2b/v \tag{7}$$

the following dimensionless parameters can be defined:

$$X = \frac{x}{(4\alpha\tau)^{1/2}}; \quad Z = \frac{z}{(4\alpha\tau)^{1/2}}; \quad \zeta = \frac{\mu}{\tau};$$

$$\xi = \frac{t}{\tau}; \quad B = \left[\frac{2b}{(4\alpha\tau)^{1/2}}\right]^2 = \frac{vb}{2\alpha} \tag{8a}$$

$$T^+(X,Z,\xi,B) = \frac{T(X,Z,\xi,B)}{q_0 2b/k} \tag{8b}$$

Note that B can be considered as the reciprocal of a Fourier number based on the dwell time τ and the strip width $2b$. Equation (6) can now be written in dimensionless form

$$T^+(X,Z,\xi,B) = \frac{(\pi B)^{-1/2}}{4} \int_0^\infty \exp\left(-\frac{Z^2}{\zeta}\right) \times \left\{\operatorname{erf}\left[(2B)^{1/2} \frac{(X/B^{1/2}+1/2)-\xi+\zeta}{(2\zeta)^{1/2}}\right] - \operatorname{erf}\left[(2B)^{1/2} \frac{(X/B^{1/2}-1/2)-\xi+\zeta}{(2\zeta)^{1/2}}\right]\right\} \frac{d\zeta}{\zeta^{1/2}}. \tag{9}$$

The exact solution of the problem defined by equations (3) is [9]

$$T(z,t) = \frac{2q_0}{k}(\alpha t)^{1/2} \cdot \operatorname{ierfc}\left[\frac{z}{(4\alpha t)^{1/2}}\right] \quad \text{for } t \leqslant \tau \tag{10a}$$

and

$$T(z,t) = \frac{2q_0}{k}\left\{(\alpha t)^{1/2} \cdot \operatorname{ierfc}\left[\frac{z}{(4\alpha t)^{1/2}}\right] - [\alpha(t-\tau)]^{1/2} \cdot \operatorname{ierfc}\left[\frac{z}{(4\alpha(t-\tau))^{1/2}}\right]\right\} \quad \text{for } t > \tau. \tag{10b}$$

Let

$$T^+(Z,\xi,B) = \frac{T(Z,\xi,B)}{q_0 2b/k} \tag{11}$$

then, by using equations (8a), the dimensionless form of equations (10) are

$$T^+(Z,\xi,B) = B^{-1/2}\xi^{1/2} \operatorname{ierfc}(Z/\xi^{1/2}) \quad \text{for } \xi \leqslant 1 \tag{12a}$$

and

$$T^+(Z,\xi,B) = B^{-1/2}\{\xi^{1/2} \operatorname{ierfc}(Z/\xi^{1/2}) - (\xi-1)^{1/2} \operatorname{ierfc}[Z/(\xi-1)^{1/2}]\} \quad \text{for } \xi > 1. \tag{12b}$$

The surface temperature profiles as a function of time are shown in Fig. 3, according to both equations (12) and (9) for $B = 1$ and 4, the temperature being normalized with reference to the maximum temperature given by the 1-D model and the time being made dimensionless with reference to the interaction time. It is worth noting that, according to the 2-D model, when the hot spot reaches a surface region the local temperature is already higher than the initial one and the cooling rate at the surface is less than the 1-D one. The maximum values of the temperature and the time at which they occur will be analysed in the next section.

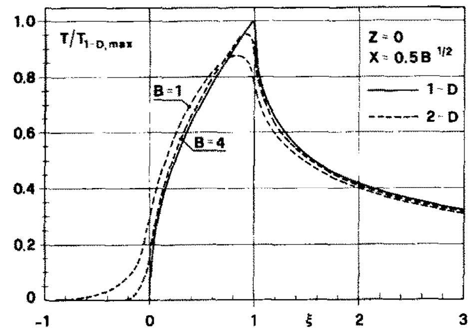

FIG. 3. Dimensionless normalized surface temperature vs dimensionless time, according to the 2- and 1-D models.

MAXIMUM TIME VALUE ANALYSIS

The maximum values of temperature as a function of time are characteristic variables of the problem under consideration. According to the 2-D model, the same values of the maximum temperature are attained, for a given Z, at different ξ and X. Thus, a meaningful comparison can be made with reference to the ratio of the maximum time values of the temperature attained respectively in the 2-D field, for a given Z and for $X = \sqrt{B}/2$ $(x = b)$, and in the 1-D field, for the same value of Z

$$R(Z,\xi,B) = \frac{T^+_{2\text{-D,max}}(\sqrt{B}/2, Z, \xi, B)}{T^+_{1\text{-D,max}}(Z,\xi,B)}. \quad (13)$$

The evaluation of the numerator of equation (13) can be performed by deriving equation (9) for $X = (B^{1/2})/2$. This leads to

$$\frac{\partial T^+(\sqrt{B}/2, Z, \xi, B)}{\partial \xi} = -\frac{1}{2\pi}\int_0^\infty \exp(-Z^2/\zeta) \times \{\exp[-B(\zeta-(\xi-1))^2/\zeta] - \exp[-B(\zeta-\xi)^2/\zeta]\}\zeta^{-1}\,d\zeta. \quad (14)$$

Setting expression (14) equal to zero yields the value of ξ corresponding to $T^+_{2\text{-D,max}}$, for given values of Z and B. By suitable transformations, as shown in the Appendix, the terms on the right-hand side of expression (14) can be put in a form similar to the left-hand side of expression (5.29) given by Oberhettinger and Badii (p. 41 in ref. [13]). The first term becomes

$$\int_0^\infty \exp(-Z^2/\zeta)\exp[-B(\zeta-(\xi-1))^2/\zeta]\zeta^{-1}d\zeta = 2\exp[2B(\xi-1)]K_0\{2[(Z^2+B(\xi-1)^2)B]^{1/2}\} \quad (15)$$

and the second

$$\int_0^\infty \exp(-Z^2/\zeta)\exp[-B(\zeta-\xi)^2/\zeta]\zeta^{-1}\,d\zeta = 2\exp(2B\xi)K_0\{2[(Z^2+B\xi^2)B]^{1/2}\}. \quad (16)$$

By introducing equations (15) and (16) into equation (14) we obtain

$$\exp(-2B)K_0\{2[(Z^2+B(1-\xi)^2)B]^{1/2}\} - K_0\{2[(Z^2+B\xi^2)B]^{1/2}\} = 0. \quad (17)$$

The roots of the transcendental equation (17) are the ξ values which, replaced into equation (9), give the maximum temperature attained for any Z for a given B.

The denominator of equation (13) is given by expression (12a) for $\xi = 1$, i.e. for $t = \tau$, obviously only for $Z = 0$, that is

$$T^+(0,1,B) = (\pi B)^{-1/2}. \quad (18)$$

For $Z > 0$, the ξ values are the roots of the equation given by La Rocca [9]

$$\tfrac{1}{2}\xi(\xi-1)\ln\frac{\xi}{\xi-1} = Z^2 \quad (19)$$

which, substituted into equation (12b), yield the maximum temperature values.

RESULTS

The roots of equations (17) and (19) were calculated by the Newton–Raphson method; the modified second kind Bessel functions, evaluated through the expressions given in Chap. 9 of ref. [14], are affected by a relative error of magnitude 10^{-9}. Maximum temperature values from equation (9), which contains non-straightforwardly integrable functions, were calculated by iterative application of Cavalieri–Simpson formulas, until the relative difference between the ith and the $(i-1)$th values reached the magnitude of 10^{-5}. The error function in equation (9) and ierfc(x) in equations (12) were evaluated by the expressions given in Chap. 7 of ref. [14]. Calculations were made for $10^{-4} \leqslant B \leqslant 10^2$ and $0 \leqslant Z \leqslant 1$.

Roots of the equations

Figure 4 shows the roots of equation (17) vs B for fixed Z. For $Z = 0$ and small B values $(O(10^{-4}))$ $\xi \to 0.5$ while for $B \to \infty$, $\xi \to 1$, that is the value of the root in $Z = 0$ for the 1-D model. Consequently, note that at the surface the maximum temperature value is attained earlier according to the 2-D model than according to the 1-D one: the smaller the B the greater is the advance. For any Z and for $B \to \infty$, obviously the 2-D roots tend to the 1-D roots. Furthermore, Fig. 4 suggests that for B values of order more than 10^2 the roots of 2-D model can be approximated with a reasonable accuracy by the simplified model. Finally, note that for a given B the greater Z the greater is the difference between the 2- and 1-D roots.

Similar considerations are suggested by Fig. 5, where for fixed B the roots of equations (17) and some roots of equation (19) are reported vs Z. One can note that the roots of equation (19) practically lie on the curve for $B = 10^2$ which, on the other hand, is nearly

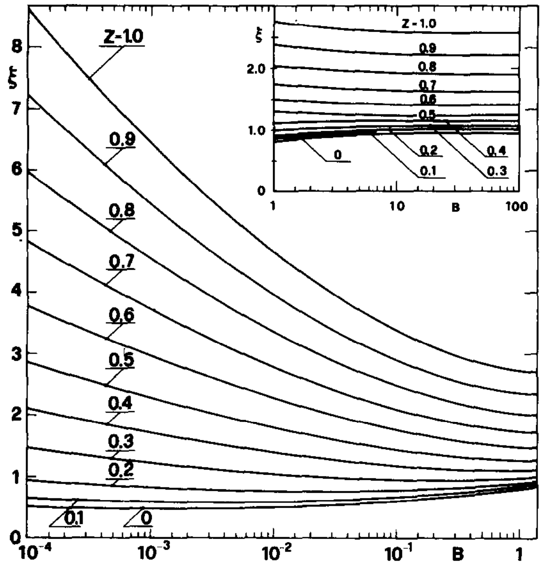

FIG. 4. Roots of equation (17): time of maximum temperature vs B at various depths for a moving strip heat source.

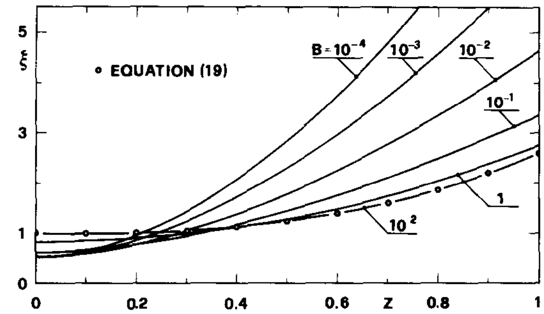

FIG. 5. Roots of equation (17): time of maximum temperature vs depth at various B values for a moving strip heat source.

coincident with that for $B = 10$. Note that, for any B, there is a particular Z for which the 1- and 2-D roots are equal. For Z greater than the above value, the maximum temperature values given by the 2-D model are reached later than those given by the 1-D model; for Z smaller than this particular value the maximum temperature values are attained earlier and the smaller the B the smaller is Z. The lead in the 2-D model is due to the component along x of the heat flux which has already perturbed the surface points when the heat source impinges on them; on the contrary, in the 2-D model the heat flux along z is less than in the 1-D model and therefore in the inner layers the maximum temperature values are reached later. The effects of the heat source velocity, v, and of its width, $2b$, on the thickness of the advance region, for a given dwell time, τ, are shown by the definition of B: for a fixed b, the smaller v (i.e. the greater τ) the smaller is the thickness of the advance region.

Table 1 gives values of roots of equations (17) in the ranges of B and Z used in the applications.

Ratio between the maximum temperature values

Figure 6 plots the ratio R defined by equation (13) vs B for fixed Z. It allows the evaluation of the error, with reference to the maximum time values of the temperature, associated to the use of the simplified model for a given B. One can remark that $R \to 1$ for $B \to \infty$, for any Z. Furthermore, the same figure shows that for $B \geqslant 3$ the use of the model for a semi-infinite body with a uniform heat flux at the surface for a time τ introduces an error the maximum value of which is 7.2% for $Z = 0.2$. For any Z, errors not greater than 5% are involved for B not smaller than 4.4; in the special case $Z = 0$, B must be not smaller than 3.9, that is a value close to that given in ref. [11]. Here $B = 3.5$ was the suggested minimum value for using the simplified model, but no inherent error was pointed out.

Table 2 provides values of R as a function of B and Z.

Figure 7 shows the ratio R vs Z for a fixed B. One can note that each curve is not at a minimum in $Z = 0$ and that all tend to unity for $Z \to \infty$. This result was to be expected as both models foresee initial temperature for $Z \to \infty$.

Figure 8 compares, at $Z = 0$, the values of R for a

Table 1. Values of dimensionless time at which maximum temperature is attained, given by equation (17)

B	$Z = 0$	$Z = 0.1$	$Z = 0.2$	$Z = 0.3$	$Z = 0.4$	$Z = 0.5$	$Z = 0.6$	$Z = 0.7$	$Z = 0.8$	$Z = 0.9$	$Z = 1.0$
1	0.83320	0.84982	0.90061	0.98810	1.11505	1.28340	1.49399	1.74685	2.04161	2.37786	2.75519
2	0.88735	0.90022	0.94030	1.01159	1.11944	1.26858	1.46183	1.70002	1.98277	2.30927	2.67855
3	0.91265	0.92389	0.95920	1.02323	1.12244	1.26302	1.44884	1.68104	1.95923	2.28229	2.64927
4	0.92776	0.93802	0.97051	1.03022	1.12436	1.25998	1.44161	1.67063	1.94645	2.26790	2.63347
5	0.93795	0.94755	0.97811	1.03490	1.12565	1.25804	1.43698	1.66398	1.93845	2.25878	2.62364
6	0.94536	0.95446	0.98361	1.03827	1.12658	1.25666	1.43374	1.65937	1.93285	2.25252	2.61692
7	0.95102	0.95973	0.98778	1.04081	1.12727	1.25564	1.43133	1.65597	1.92875	2.24796	2.61204
8	0.95550	0.96390	0.99106	1.04279	1.12780	1.25483	1.42948	1.65339	1.92563	2.24449	2.60832
9	0.95915	0.96729	0.99372	1.04439	1.12822	1.25419	1.42800	1.65132	1.92316	2.24175	2.60541
10	0.96219	0.97011	0.99592	1.04569	1.12856	1.25365	1.42680	1.64964	1.92116	2.23954	2.60305

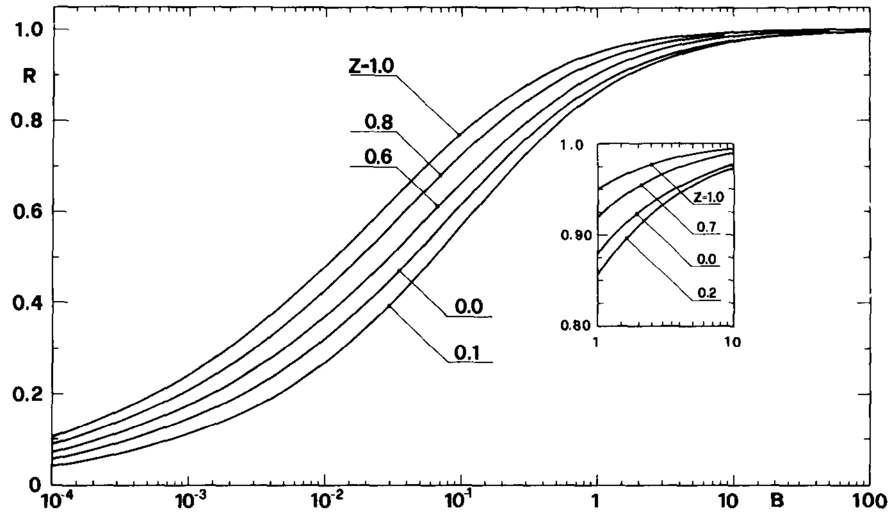

FIG. 6. Ratio of 2- to 1-D maximum temperatures vs B at various depths.

Table 2. Ratio of 2- to 1-D maximum temperatures, given by equation (13)

B	$Z = 0$	$Z = 0.1$	$Z = 0.2$	$Z = 0.3$	$Z = 0.4$	$Z = 0.5$	$Z = 0.6$	$Z = 0.7$	$Z = 0.8$	$Z = 0.9$	$Z = 1.0$
1	0.87901	0.86216	0.85583	0.86008	0.87223	0.88829	0.90476	0.91965	0.93226	0.94264	0.95109
2	0.92345	0.91306	0.90963	0.91311	0.92175	0.93280	0.94386	0.95358	0.96159	0.96800	0.97309
3	0.94243	0.93445	0.93275	0.93589	0.94284	0.95145	0.95989	0.96717	0.97307	0.97774	0.98139
4	0.95328	0.94730	0.94595	0.94886	0.95475	0.96187	0.96872	0.97456	0.97925	0.98291	0.98577
5	0.96039	0.95546	0.95459	0.95731	0.96247	0.96856	0.97434	0.97922	0.98311	0.98613	0.98848
6	0.96546	0.96127	0.96072	0.96322	0.96790	0.97323	0.97824	0.98243	0.98576	0.98833	0.99032
7	0.96928	0.96565	0.96533	0.96776	0.97193	0.97668	0.98110	0.98478	0.98768	0.98993	0.99165
8	0.97236	0.96907	0.96892	0.97123	0.97505	0.97933	0.98329	0.98658	0.98915	0.99114	0.99266
9	0.97468	0.97182	0.97181	0.97401	0.97754	0.98144	0.98503	0.98799	0.99031	0.99209	0.99345
10	0.97667	0.97410	0.97419	0.97629	0.97957	0.98316	0.98644	0.98913	0.99124	0.99285	0.99409

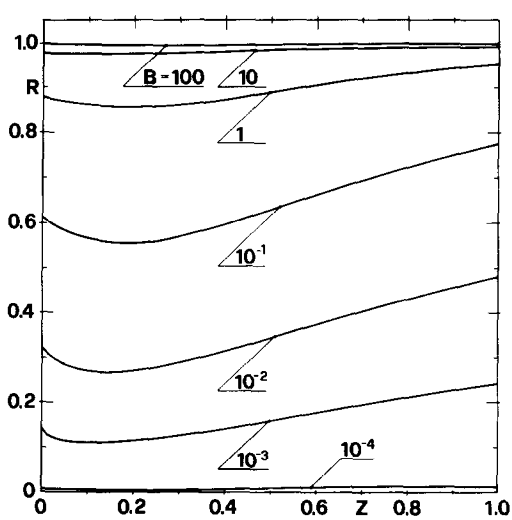

FIG. 7. Ratio of 2- to 1-D maximum temperatures vs Z at various B values.

stationary heat source, the solution of which is given by Carslaw and Jaeger on p. 264 in ref. [2], and for a moving source. Note that at the surface the stationary source solution is approximated by the 1-D model with an error not greater than 5% for $B > 3.2$. The same indication was given also by Beck [15] for a different source geometry.

CONCLUSIONS

The exact solution to the quasi-steady state two-dimensional temperature distribution in a semi-infinite solid for a uniform strip heat source moving at a constant velocity along its surface has been analysed in detail. The maximum time temperature as a function of the depth and the time at which it is attained were analytically treated. The solution to the above problem, which is particularly relevant in laser and electron beam surface processing, can be approxi-

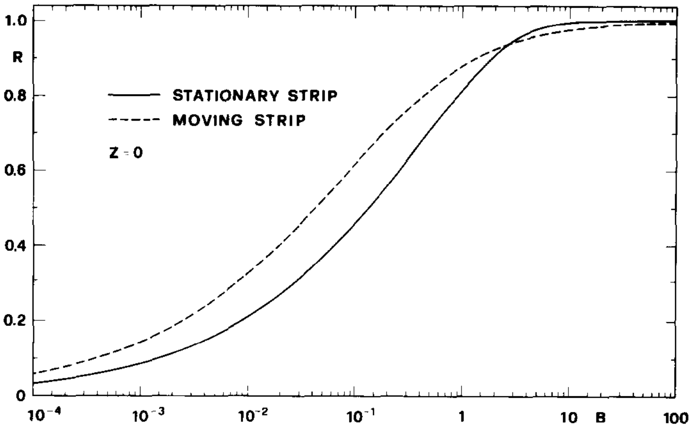

FIG. 8. Ratios of 2- to 1-D maximum temperatures for stationary and moving heat sources vs B at the surface ($Z = 0$).

mated by the 1-D semi-infinite model for fast and wide sources.

A comparison of the two solutions has been made in terms of the two dimensionless numbers B and Z. Results are reported both in diagrams and in tables. They show that at the surface ($Z = 0$) the maximum temperature error related to the 1-D solution is less than 5% for $B > 3.9$. Nevertheless, even for $B = 2$ the error reaches 9%. It slightly increases with depth for Z values of practical interest ($Z < 0.3$). The maximum temperature in the 2-D solution is reached earlier or later than in the 1-D solution, depending on the depth, for a given B.

A comparison is also made with the 2-D solution for a stationary strip source. Both 2-D solutions, stationary and moving, are well approximated by the 1-D one for dwell Fourier numbers the magnitude of which is smaller than 10^{-2}.

REFERENCES

1. J. C. Jaeger, Moving sources of heat and temperature at sliding contacts, *Proc. R. Soc. New South Wales* **76**, 203–224 (1942).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd Edn. Oxford University Press, Oxford (1959).
3. D. Rosenthal, The theory of moving sources of heat and its application to metal treatments, *Trans. Am. Soc. Mech. Engrs* **68**, 849–866 (1946).
4. N. R. Des Ruisseaux and R. D. Zerkle, Temperature in semi-infinite and cylindrical bodies subjected to moving heat sources and surface cooling, *J. Heat Transfer* **92C**, 456–464 (1970).
5. H. E. Cline and T. R. Anthony, Heat treating and melting material with a scanning laser or electron beam, *J. Appl. Phys.* **48**, 3895–3900 (1977).
6. I. Chen and S. Lee, Transient temperature profiles in solids heated with scanning laser, *J. Appl. Phys.* **54**, 1062–1066 (1983).
7. D. J. Sanders, Temperature distributions produced by scanning Gaussian laser beams, *Appl. Optics* **23**, 30–35 (1984).
8. Emploi de fasceaux d'électrons commandés par mini-calculateurs dans le traitment thermique superficiel des métaux, *Cah. Tech. Sciaky*, No. 11 Fevrier, Vitry sur Seine (1980).
9. A. V. La Rocca, Models of thermal fields with high entering flux for laser processing, presented at 5th Int. Symp. on Gas Flow and Chem. Lasers, Oxford, U.K. (1984).
10. M. Lax, Temperature rise induced by a laser beam, *J. Appl. Phys.* **48**, 3919–3924 (1977).
11. O. Sandven, Laser surface transformation hardening. In *Metal Handbook*, Vol. 4, pp. 507–521, 9th Edn. Am. Soc. Met., Metals Park, Ohio (1981).
12. P. Munari, Studio del comportamento termico dei materiali sottoposti a trattamento termico mediante laser con modello a parametri costanti. *Progetto Finalizzato CNR: Laser di potenza.* RTM, Vico Canavese (1981).
13. F. Oberhettinger and L. Badii, *Tables of Laplace Transforms.* Springer, New York (1973).
14. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Math. Ser., Vol. 55. National Bureau of Standards, Washington, D.C. (1964).
15. J. V. Beck, Transient temperatures in a semi-infinite cylinder heated by a disk source, *Int. J. Heat Mass Transfer* **24**, 1631–1639 (1981).

APPENDIX

The first term on the right-hand side of equation (14) can be written as

$$\int_0^\infty \frac{1}{\zeta}\exp(-Z^2/\zeta)\exp\{[\zeta^2-2\zeta(\xi-1)+(\xi-1)^2]B/\zeta\}\,d\zeta$$

$$= \int_0^\infty \frac{1}{\zeta}\exp(-B\zeta)\exp\{-[Z^2+B(\xi-1)^2]/\zeta+2B(\xi-1)\}\,d\zeta. \quad (A1)$$

Let

$$\gamma = Z^2+B(\xi-1)^2;\quad \beta = 2B(\xi-1) \quad (A2)$$

by substituting equations (A2) into equation (A1) and remembering expression (5.29) on p. 41 of ref. [13], one obtains

$$\exp(\beta)\int_0^\infty \frac{1}{\zeta}\exp(-B\zeta)\exp(-\gamma/\zeta)\,d\zeta = \exp(\beta)2K_0[2(\gamma B)^{1/2}] \quad (A3)$$

which yields expression (15).

Expression (16) can be obtained in a similar manner.

COMPARAISON ENTRE DEUX MODELES DU CHAMP THERMIQUE DANS LES PROCEDES DE TRAITEMENT THERMIQUE SUPERFICIEL AVEC LASER ET FAISCEAUX D'ELECTRONS

Résumé—Dans les procédés de traitement thermique superficiel avec laser et faisceaux d'électrons la connaissance du champ des températures est trés important et, en particulier, celle du maximum de température atteint. On analyse la solution pour le cas d'une source chaude uniforme en forme de bande mobile à vitesse constante le long de la surface d'un corps semi-infini. Les auteurs présentent une comparaison du modèle approché unidimensionnel et du modèle bidimensionnel. On donne les rapports entre les valeurs des températures maximales déduites des deux modèles et entre les temps où elles sont atteintes. On en déduit ainsi des indications pour évaluer le domaine des variables dans lequel les résultats atteints par les deux modèles s'accordent à peu près.

EIN VERGLEICH ZWISCHEN MODELLEN VON MIT LASER-ODER ELEKTRONENSTRAHL BEHANDELTEN TEMPERATURFELDERN

Zusammenfassung—Die Kentnissen der termischen Feldes und im besonderem der höchsten erreichbaren Temperatur stellen bei Laser-oder Elektronenstrahl Oberflächenbehandlung ein sehr wichtiges Problem dar. Die zweidimensionale Lösung einer gleichförmigen und strichförmigen Wärmequelle die sich mit konstanter Geschwindigkeit auf der Oberfläche eines halbunendlichen Körpers bewegt, wird analysiert. Ein Vergleich zwischen der eindimensionalen Näherung und dem halbunendlichen zweidimensionalen Modell, wird beschrieben. Das Verhältnis zwischen den höchst erreichen Temperaturwerten und derer dem Modell nach erforderlichen Bestrahlungszeiten, wird vorgezeigt. Richtlinen zur Bestimmung des Parameterbereichs in dem die von beiden Modellen erholtenen Ergebnisse vergleichbar sind, werden angegeben.

СРАВНЕНИЕ МОДЕЛЕЙ РАСЧЕТА ТЕМПЕРАТУРНЫХ ПОЛЕЙ ПРИ ОБРАБОТКЕ ПОВЕРХНОСТЕЙ ЛАЗЕРНЫМ И ЭЛЕКТРОННЫМ ЛУЧАМИ

Аннотация—При обработке поверхностей лазерным и электронным лучами важно знать распределение температуры и максимально возможные ее значения. Анализируется двумерное решение для однородного источника тепла, движущегося с постоянной скоростью на поверхности полубесконечного тела. Представлено сравнение приближенных одномерной и двумерной моделей. Приведено отношение максимальных температур, а также время их достижения для этих двух моделей. Даны оценки диапазона изменения переменных, в пределах которого результаты, полученные для двух моделей, хорошо согласуются.